similar triangles POA, OQ_1P_1 we have $PO:OA::OQ_1:Q_1P_1$. But $PO=1$ and hence Q_1P_1=the area of the rectangle $OQ_1A''A$. Similarly, it may be shown that Q_3P_3=the sum of the areas of the rectangles OQ, $A''A$ and $Q_1Q_3B'''B''$, while XP_4=the latter sum plus the area of the rectangle Q_3XCC'''.

Greater accuracy may frequently be obtained by dividing each of the arcs AB, BC into several portions (instead of two) such that any one divides the area of its circumscribing rectangle into two equal portions. Again, the selection of a relatively large number of such points and the completion of the subsequent constructions evidently leads to a curve analagous to $OP_1P_2P_3P_4$ but representing a closer approximation to the true integral curve. Thus, the integral curve itself may be sketched in with considerable accuracy, thereby affording a means of computing instantly the area under the curve between any two given ordinates.

We shall not dwell upon the generality of this method of integration or upon various other features of it which are revealed by a close examination. It is to be remarked, however, that many important two dimensional determinations such as areas, centers of gravity, and moments of inertia, are admirably carried out by similar methods. Increased familiarity with the graphical calculus, in fact, always convinces one of its merits, and, as is natural, there are some who would even abandon quite completely the standard form of calculus in favor of the graphical. While it will be seen from what I have to say later on, that such an extreme in no wise seems desirable to me, it may be instructive to note the attitude of the graphical enthusiasts as illustrated in the following quotation taken from the introduction of a recent work.*

"All up to date teachers of engineering and applied science generally now recognize the vast superiority of the graphical over the purely mathematical methods of instruction of almost every description. . . . The attempt to employ purely mathematical, in preference to graphical, methods seems to me quite as absurd as attempting to teach geography by giving the position of towns in terms of their latitude and longitude and explaining the shape of a country by giving the equation of its coast-line instead of by employing the graphical method, that is, exhibiting a map."

The opinions here expressed certainly furnish food for thought. It is undeniably true that the curves with which one deals to-day in applied science are rarely if ever given by means of their equations, but rather graphically. They are determined experimentally by ascertaining a sequence of points satisfying prescribed conditions, and then drawing a smooth curve through them. It is the inability of the ordinary calculus to actually furnish the area under a curve when given in this manner, or to determine centers of gravity and moments of inertia for figures thus bounded—in short, to really *do* things—that is its fatal weakness for the engineer and physicist.

Graphical Calculus, by A. H. Barker. London, Longmans, 1905.

Enough has now been said to justify the following inquiry. In view of the recognized importance of the so-called graphical calculus in applied science and in view also of the fact that a large and increasing number of pupils who study the calculus to-day have technical careers in view, would it not be well for the calculus in its present scholastic form to recognize in larger measure the merits of the graphical method and incorporate some of its ideas? My own belief, as already indicated, is that such departures would be advisable. This does not imply an undervaluation of the logical aspect of the calculus but rather a proper valuation of the newer methods and processes which have come to play so important a rôle in it. Neither does it imply a revolution in our methods of instruction, but merely a new point of view in which present conditions, especially as regards the needs of the majority of our pupils, are more squarely met. But to what extent, it may be asked, do you really mean to change the ordinary first course? Would you omit some of the present material in order to make room for the graphical? Doubtless, all would be willing to introduce some of the latter *if there time for it.* Of this I shall have more to say in my second thought, mentioned at the beginning and which I now proceed to consider.

All mathematicians recognize in their scientific work the value of the process of successive approximation. I would now ask whether this process should not be better recognized also on the pedagogical side. In the teaching of the calculus, for example, can we after all expect the average pupil in his first course to appreciate the modern point of view as regards the infinitesimal? True it is that clearness cannot be assured to the scientific mind without such refinements, but should we expect to make the Calculus clear in this sense to a beginner? However heterodox I may appear, I do not believe so. Such insight, in fact, belongs usually to later stages in one's development. The question which the beginner desires answered is, what is the calculus about *about*—what does it *do?* The attitude is the same for any branch of scientific study when first approached and it is only after such questions have been answered that there is a natural inquiry into or need for the really *exact.* Not until then is it time for the further approximations to that end. Moreover, many who are studying the Calculus to-day —in fact the large majority—never reach this second stage at all for lack of natural endowment, yet they have a perfect right to study the subject and are capable of deriving great profit from it. The distinctly higher approximations, leading ultimately into the modern theory of functions of a real variable, are, of course. for those only who possess primarily theoretical inclinations. I believe, therefore, that a more thoughtful application of the successive approximation principle to our teaching would improve it. Likewise, our texts might perhaps be bettered in this particular. They, likewise, should be written in the knowledge that the average beginner is concerned with facts, not with reasons why. Thus it may be said that while our newer texts have indeed brought the calculus nearer home to

the average pupil through the introduction of problems based upon daily experience or upon the experiments of the laboratory, still more will stand to the credit of our time when a different arrangement of material has been produced — one, in fact, which will better recognize the successive approximation principle on the pedagogical side. To be more specific, let us be satisfied in the beginning with a rough definition of the derivative, allowing intuition and not rigor to be the chief factor in supplying it and therefore drawing very largely upon the geometrical side. Included in this first approximation should be some simple problems in maxima and minima and the first notions of the integral calculus, all following the same general plan of putting emphasis upon the "what" and *not* the "why." This may be done within the first few weeks of the course after which the second approximation is to be entered upon. In this the whole field should be enlarged and strengthened logically as far as time will permit. It is in this latter connection that opportunity should be found to introduce something at least of the graphical methods as outlined above. Depending upon whether the class is primarily of technical students or otherwise, the teacher should use his judgment as to where the chief emphasis in this second approximation should fall. Conceivably, there are classes in which one might eventually develop the notion of rigor, even in a first course, to such an extent as to acquaint the student with the meaning of an arithmetic *epsilon* proof, while there are classes in which the graphical should largely predominate at the expense of the analytical. Neither element can be omitted, but more judgment should be used in proportioning them.

In recognizing, therefore, the great advance which has been made in later years toward a correct teaching of the Calculus, we would merely point out two features in which still further improvement seems possible. However suicidal it may seem to the analyst, the first course in calculus should be framed to meet more adequately the mental endowments of the average pupil. In particular, we should give a larger recognition to the merits of the so-called graphical calculus. We should also recognize more fully that it is the facts of the calculus and not the logical coherence of it that appeal primarily to the beginner, and this in turn necessitates a fuller recognition on the pedagogical side of the principle of successive approximation.

A NEW THEOREM IN THE GEOMETRY OF CONICS.*

By ALAN SPENCER HAWKESWORTH.

If two or more triangles, to which any conic is in common escribed, have each a vertex upon one and the same axis of said conic, and if, in each case, a circle be described, passing through the other two vertices of the said triangle, and with its center upon the said axis:

Then will the resultant system of circles have the other axis of the conic for their common radical axis, and will pass in common through the two foci, real or imaginary, as the case may be, lying upon that axis, while their "limiting points" upon the first and given axis, and line of centers of the said system of coaxal circles, will be, in turn, the imaginary or real foci of the conic.

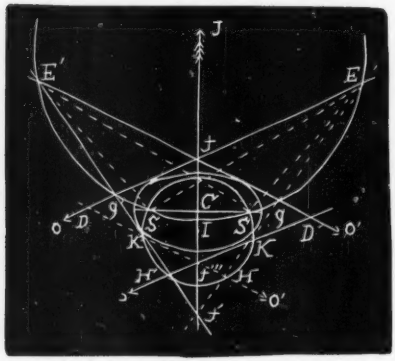


Fig. 1.

(A) For, first, take any central conic (Fig. 1), and let Et, Et' be any two tangents to it, concurring in E, and cutting its minor axis in points t and t', respectively. Draw $E't$ and $E't'$, two other tangents, symmetric about the minor axis with Et and Et', and thus concurring with them in t, t', respectively upon the axis. And let Egt' and $E'tg$ cut in g; and $E'g't'$ and Etg' cut in g', forming a pair of congruent triangles Etg and $E'tg'$, to which the conic is escribed; with a common apex t upon that conic's minor axis. Join S and S' the conic's two foci, to points E and E', in lines SE, SE', $S'E$ and $S'E'$.

Then, by a well known theorem, and by symmetry,

$$\text{angles } SEg = S'E'g' = S'Eg' = SE'g.$$

So that points $EgSS'g'E'$ are concyclic about a center lying on tt' their axis of symmetry, the conic's minor axis.

And in like manner, for any other two pairs of tangents, symmetric about that minor axis. Say, for example, $t'HK$ and $t'H'K'$; $Ht''K$ and $H't''K$; concurring upon the minor axis in $t't''$, and cutting in KK'. Whence as before,

$$\text{angles } SHK = S'H'K = S'HK' = SH'K,$$

and thus $HKSS'K'H'$ are concyclic about a center lying on the minor axis.

*Read before the American Mathematical Society, September, 1909.

So that, conversely, if from any point t upon the minor axis of a conic, two tangents Etg' and $E'tg$ be drawn, and these be cut in E, g, respectively, by any third tangent to the curve, forming thus a triangle, with a vertex t upon the minor axis, to which the said conic is escribed, then the circle, whose center is on the minor axis, and which passes through the other two vertices E, g of said triangle, will also pass through the conic's two foci S, S'.

And hence, if two or more such triangles be drawn, as for example, Etg and $Kt''H$, then all such circles $EgSS'$ and $HKSS'$, passing in common through the two foci, have the conic's major axis for their common radical axis. While C, the conic's center, will be the position both of its imaginary foci, and of the now imaginary common "limiting points" of the said system of circles.

(B) Secondly, taking now any conic, ellipse, hyperbola, or parabola (Fig. 2), again let us have any two tangents ET and ET', concurring in E, but now cutting the conic's major axis in points T and T', respectively. Once more let us draw two fresh tangents $E'T$ and $E'T'$, but symmetric now about the major axis to ET and ET', and thus concurring with them upon it in T and T', respectively. And let them cut ETg' and $E'g'T'$ in g', and EgT' and $E'Tg$ in g; thus forming a symmetric pair of congruent triangles ETg and $E'Tg'$, to which the conic is escribed, each having a common vertex T upon that conic's major axis.

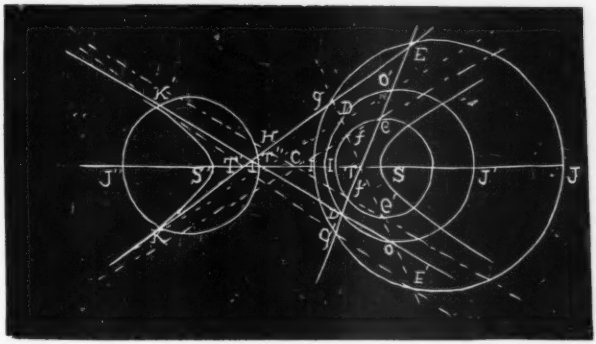


Fig. 2.

Let I and J be the two common excenters of the said two congruent triangles upon the conic's major axis, their axis of symmetry. Then IgJ, $Ig'J$, IEJ, and $IE'J$ being right angles, points I, g, E, J, E', g' are concyclic about the medial point of IJ upon the said major axis. And if the conic's foci S, S' be joined to, say E, in lines SE, $S'E'$, then, as before, $SEg'=S'Eg$; so that $SEI=S'EI$, and $SI:S'I=SE:S'E=SJ:S'J$ (Euclid VI, 3 and A).

Conversely, therefore, the circle through Eg, whose center is on TT', the conic's major axis, ever harmonically divides the inter-focal distance SS' in points I and J. Or $CI:CS=CS:CJ$, whatever may be the triangle ETg, to which the conic is escribed, and which has its vertex T upon that conic's major axis.

Thus, for example, if $KT'H'$ (Fig. 2) be such a triangle, and the circle through KH', whose center is on the major axis, cuts that axis in points I', J', then, as before, $CI':CS'=CS':CJ'$.

From this it follows that the given conic's minor axis is the common radical axis of all such circles, and their common "limiting points" are the two foci of the conic. While the imaginary foci, lying at that conic's center, are now also the imaginary common intersections of that minor axis by any and all the circles.

Corollary 1. Hence when the conic is a circle, CS being now infinitesimal, all such circles must pass through its center. This fact is also evident from the equality of the angles CET and CEg; so that C is ever the common excenter of all the triangles escribed to the circle which has it for its center.

Corollary 2. On the other hand, with a parabola, any and all such circles must be concentric about the focus S, a fact which can also be shown as follows: The external angle ETg of the isosceles triangle ETE' is double either of the two equal interior angles TEE' or $TE'E$. And hence the center of the circle through $EE'g'g$ must lie concyclic to the points E, T, g (Euclid III, 20), while, by symmetry, it must also lie on axis TS. And next, by a well known theorem, the circumcircle of the circumscribing triangle ETg must pass through S, the focus of the parabola. From which it follows that the center of the circle through $EE'g'g$ must be either T or S. Now it cannot be T, since Tg and TE being equally inclined to the axis, the tangent EgT', by the nature of the parabola, must be so inclined that it cuts TE at a greater distance from that axis and from T than it cuts Tg. Hence S must be the center desired. And in like manner it can be shown that S is also the center of the circles through $KH'HK'$, and through $ODD'O'$, etc.

Corollary 3. So that were any circumscribing triangle DEH to an unknown conic given, and any transversal $tt't''$ or $TT'T''$ of that triangle posited as the position of an unspecified axis of unknown magnitude of the said unspecified conic, then there is ever an unique solution (with the exception yet to be noted in Corollary 4, following) of one, and one conic only, which shall be inscribed or escribed, as the case may be, to the given triangle, and have the given transversal for an axis, major, or minor, as the case may be.

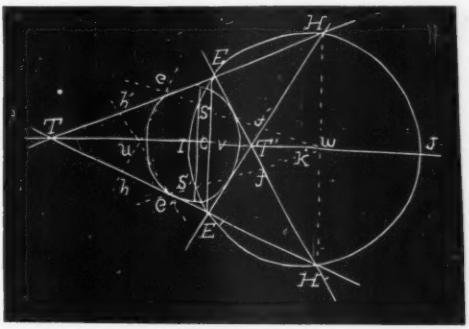
For describing a second circumscribing and congruent triangle $D'E'H'$, symmetric about the given axis, and cutting the first triangle in points t, t', t''; or T, T', T''; g, g'; K, K'; and O, O', we obtain three pairs of triangles ETg and $E'Tg'$, HTK and $H'T'K'$, DTO and $D'TO'$, each with a vertex upon the given axis. Wherefore, by the foregoing theorem, the circles through $Egg'E'$, $HKK'H'$, and $DOO'D'$ must determine the other axis; cutting, and passing in common through the foci, when the given axis happens to be the conic's minor, but none cutting, when the given axis happens to be a major, in which latter case the external radical axis, and its two real "limiting points" give, as stated, the minor axis and foci desired.

And the foci being thus found, a perpendicular from either upon any one of the six known tangents will give us, by an elementary theorem, a point on the conic's auxiliary circle, and hence the magnitude of the major axis. So that the desired conic can now be readily drawn.

Corollary 4. But an exception to the foregoing Corollary 3 obtains when the said transversal happens to pass through a vertex of the given circumscribing triangle.

For, if it pass through a vertex, without also bisecting the angle, then the corresponding two sides of the resultant symmetric triangle form in the said vertex four concurrent rays. And no conic is possible, other than in the degenerated form of a right line, coinciding with the given axis.

While if the given axis bisects, internally, or externally, an angle of the given circumscribing triangle, then the corresponding two sides of the symmetric triangle coalesce with the two sides of the given triangle, and a symmetric quadrilateral $HE'H'ET'T$ is formed (Fig. 3). An identical figure being given, whether the said axis bisects internally the common vertical angle HTH' of the two symmetric and congruent triangles ETH and $E'TH'$, or bisects externally in common the vertical angles $E'TH$ and ETH' of the two symmetric and congruent triangles $E'TH$ and ETH'.



Hence the conic is as yet entirely indeterminate, without further data. For we now have but one pair of symmetric and congruent triangles $E'T'H$ and $ET'H'$, having a common vertex T' upon the given axis, and thus but one circle through the points E', H, J, H', E, I, upon which the desired conic's foci, real, or imaginary, must lie. And innumerable conics are thus possible, having the given axis for either a major or a minor.

But if, in addition, we are given the position of C, the center of the required conic; then, if C falls within the limits I, J, the two excenters upon TT' of $E'TH$ and $ET'H'$, a perpendicular through it will cut the circle through $IE'HJH'E$ in S, S', the desired foci, and TT' will thus be the minor axis of the desired unique conic.

While, with C falling anywhere outside IJ along TT', then the perpendicular through it will now be the minor axis of the required unique conic, whose major is TT'. And the fixed ratio $CI \cdot CJ = CS^2 = CS'^2$ gives us the desired foci.

Furthermore, we can point out the limits of C, with regard to any desired conic, as follows.

Join $E'E$ and HH', cutting, and being perpendicularly bisected by TT' in points v and w, respectively. Let u be the medial point of magnitude TT', and let het, $h'e't'$ be the symmetric and congruent medial triangles of HET and $H'E'T'$, respectively.

Then, obviously, he and $h'e'$ bisect in common TT' in u; and, in like manner, hvt and $h'vt'$ concur in v, and etw and $e't'w$ concur in w, and not otherwise. Were the symmetric medial triangles of $E'T'H$ and $ET'H'$ taken, their produced sides must likewise concur in the said three points u, v, w, along TT'.

Lastly, let K be the center of the circle through points I, E', H, J, H', E. We will show this to be also the common intersection of TT' by the circumcircles of ETH, $E'TH'$, $E'T'H$, and $ET'H'$.

(a) Then, with the given center C at either I or J, there evidently results either the common incircle or excircle of ETH and $E'TH'$, or one of the two common excircles of $E'T'H$ and $ET'H'$, as the case may be.

(b) With C at either v or w we have, in each case, a degenerated right line conic, having foci at E', E or H, H', as the case may be; each of which may be considered as the extreme form of, or limit between, an ellipse and a hyperbola.

(c) The center C falling anywhere between I and v gives us, as shown, foci falling symmetrically upon the arc $E'IE$, so that we now have an ellipse, inscribed to ETH and $E'TH'$, or escribed to $E'T'H$ and $ET'H'$, with TT' as its minor axis; while, in like manner, C falling anywhere between w and J gives an ellipse, with TT' as its minor axis, and foci which fall symmetrically upon arc HJH', escribed in common to ETH and $E'TH'$, or to $E'T'H$ and $ET'H'$.

(d) And similarly, C falling anywhere between v, w gives us a hyperbola, whose minor axis is TT', escribed in common to ETH and $E'TH'$, or to $E'T'H$ and $ET'H'$, having foci upon the symmetric arcs $E'H$ and EH', and the asymptotal angles which rise from zero, as C moves from v to T', attain the maximum value of $E'T'H$ when C is at T', where $ET'H$ and $E'TH$ are the asymptotes; and sink again towards zero as C moves from T' to w; since, by the law of the hyperbola, real tangents concurring upon the minor axis, and thus touching opposite branches of the curve, must ever concur in an angle less than the supplementary asymptotal angle (this being their ideal maximum), and hence must concur supplementarily in an angle ever greater than the said asymptotal angle.

(e) Next, taking C at u, the medial point of TT', we plainly obtain, once more, a degenerated ellipse or hyperbola, a right line conic, but now one coincident with TT', its major axis, and having foci at T, T'; a fact also evident from the consideration that $IT':IT=JT':JT$, or $uI'uJ=uT^2=uT'^2$.

(f) From this it follows that if C be taken anywhere within the limits u and I, then the foci S, S' must fall, one between T, I and the other between T', I, to fulfill our ratio $CI'CJ=CS^2=CS'^2$, so that we now obtain an ellipse, inscribed to ETH and $E'TH'$, or escribed to $E'T'H$ and $ET'H$, and having TT' for its major axis.

(g) While, in like manner, C falling anywhere along TT', produced beyond J, gives us an ellipse, escribed in common to ETH and $E'TH'$, or to

ETH and $ET'H'$, having TT' for its major axis, and foci which fall between J, K [as we will later prove], and upon TT' produced beyond J respectively.

(h) On the other hand, if C be taken along TT' produced, anywhere beyond u, the same ratio $CI \cdot CJ = CS^2 = CS'^2$ necessitates that its two foci S, S' must fall outside the limits T, T', along TT' produced in both directions, so that now we obtain a hyperbola, escribed in common to ETH and $E'TH'$, or to $E'T'H$ and $ET'H'$, having TT' for its major axis; and an asymptotal angle, which rises from zero, as C moves from u towards T, attains its maximum value of HTH' when C is at T, and HT, $H'T$ are the asymptotes; and sinks again towards zero as C retreats from T along TT' produced towards infinity. For obviously, as before stated, by the law of the hyperbola, the concurrent angle HTH' of the tangents TH and TH', which touch the same branch of the curve, must ever be greater than their ideal asymptotal angle.

(i) Lastly, escribing a parabola in common to ETH and $E'TH'$, or to $E'T'H$ and $ET'H'$, having TT' for its major axis, then, in either case, its focus S, by a well known theorem, must lie on the circumcircles of the said circumscribing triangles ETH and $E'TH'$, or $E'T'H$ and $ET'H'$; while by the foregoing Corollary 2, this focus S must also be the center of the circle through $E'HEH'$, and hence K, this center, is also the common intersection of axis TT', as stated, by the four circumcircles of ETH, $E'TH'$, $E'T'H$ and $ET'H'$. And thus, when the center of our conic has retreated to infinity in either direction along TT', then K will be the position of the focus of the resultant parabola, escribed to ETH and $E'TH'$, or to $E'TH$ and $ET'H'$.

(h) So that, returning to the previously mentioned hyperbola, whose center is beyond u, along TT' produced, and whose major axis is TT', it further follows that one of its foci must ever fall between the limits T', K, and thus within the circumcircles of its circumscribing triangles ETH and $E'TH'$, or $E'T'H$ and $ET'H'$; and similarly, as stated, one focus of the ellipse, escribed to the same triangles, must fall within the limits K, J and outside those circumcircles. Results independently proved in a previous theorem of mine (*Supplemento ai Rendiconti del Circolo Matematico di Palermo*).

Corollary 5. Wherefore, were we given any two tangents to an unspecified conic, Et and Et', or ET and ET', concurring in E, and were also given the position of the said conic's two axes, and thus its center, but knew neither the magnitude, nor the names of the said axes, then in every case (with the exceptions yet to be noted, in Corollary 6) a unique solution is given by the foregoing theorem, that is, one conic and one only, fulfilling the required conditions of touching the two given tangents, and having its axes in the given positions.

For, letting Et and Et' (Fig. 1) or ET and ET' (Fig. 2) be our two given tangents, with tCt' and ACA', or TCT' and BCB', as the two unspecified axes, we can, obviously, take either of these two axes as the axis of symmetry for our second pair of tangents $E't$ and $E't'$, or $E'T$ and $E'T'$, cutting

our first pair in t, g and g', t', or in T, g and g', T', respectively; and giving us then two congruent and symmetric triangles Etg and $E'tg'$, or ETg and $E'Tg'$, whose vertical angles at t or T are externally bisected by the said axis of symmetry tt' or TT'. From which it follows, as shown, that the circle through $EgIg'E'J$ either cuts the other axis ACA' in S, S', the foci of the required conic; or else, if it cuts that other axis merely in imaginary points, then the ratio $CI \cdot CJ = CS^2 = CS'^2$ gives us these foci.

And note: if the said circle through $EgIg'E'J$ has its center on the conic's major axis, then C is external to IJ; while if its center is on the minor, then C is within IJ; and hence, were we to draw EI, or EJ, the internal or external bisector of the concurrent angle tEt' or TET', according to whether C is nearer to I or to J, then the said bisector EI or EJ will ever cut and cross the major axis of our conic before it can cut the minor in I or J, as the case may be.

So that, conversely, with the given concurrent tangents Et and Et', or ET and ET', and the two unspecified axes, if the concurrent angle tEt' or TET' be bisected, internally, or externally, by EI or EJ, as the case may be, then this bisector will ever cut and cross first that axis which is to be the major of our unknown conic, an important fact to know.

But, if the said bisector should happen to pass through C, and cut neither axis first, then the conic must thereby be the incircle or excircle of tEt' or TET', according to whether the said bisector is the internal or external bisector of tEt' or TET'.

Lastly: with C as a known point, the desired conic is plainly central. But were one of the two axes and C specified as lying at infinity, then we must evidently have a parabola, and the medial point K of the points I and J wherein our internal and external bisectors of TET' cut our known axis TT', will be its focus, and the parabola can thus be drawn.

Oorollary 6. But, as in Corollary 4, an exception to the foregoing evidently occurs in the special case of the concurrent point of our given two tangents falling upon one of our given axes.

For if the said axis does not bisect the concurrent angle, then we merely obtain, by axial symmetry, two sets of concurrent rays, and no conic is possible, other than in the degenerated form of the said axis itself.

While, with the axis bisecting internally or exteraally the concurrent angle, then the twofold axial symmetry gives us a rhombus* within which we can plainly place an incircle and innumerable inscribed ellipses, each having either of the two axes for its major, while either axis can also be the major axis of innumerable escribed hyperbolas, whose asymptotal angles vary from zero, up to an ideal maximum, that they can never reach, equal to that of the rhombic angles which lie on the chosen major axis.

Corollary 7. In the theorem, the circle whose diameter is the "limit-

*So that our conic is here indeterminate. And further data are needed.

ing point," and thus the inter-focal distance of the real or imaginary foci, as the case may be, plainly cuts all the co-axial circles orthogonally.

Corollary 8. In the theorem, the hyperbola's asymptotes being also tangents, the circle through the points (say e, f, e', f') in which any two tangents cut them, is also one of the co-axial system having the minor axis for its radical axis. So that $Ce.Cf=CS^2=Ce'.Cf'$.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

NOTE ON SHORT METHODS IN ARITHMETICAL CALCULATIONS.

From time to time, we have sent to us for publication, "short cuts" and "lightning methods." Most of these are of no theoretical and little practical value, and hence cannot be given a place in the MONTHLY.

The short cuts below were sent to us by Mr. Charles H. Case of Chicago, who is now nearly 81 years old. He says the list, consisting of eighteen examples, was prepared for the students of Wheaton College, November, 1896. We publish a few of them because we have found some of them useful in practical computation, having used them for years.

Mr. Case says, "The examples given should be wrought without the use of more figures than are used in the same." The principles used may be found mainly in the algebraic formulae given below.

$$(a\div b)(b\div a)=1;\quad (a+b)^2=a^2+2ab+b^2;\quad (a+b)(a-b)=a^2-b^2).$$

1. $(5\frac{1}{2})^2=30\frac{1}{4}$; $(7\frac{1}{2})^2=56\frac{1}{4}$; $(65)^2=4225$; $(88)^2=7744$; $96^2=9216$; $36^2=1296$; $76^2=5776$.

2. $625^2=\begin{array}{c}360625\\ \hline 3\\ \hline 390625\end{array}$; $876^2=767376$; $2496^2=6230016$.

3. $99649964125^2=9928129699281296015625$
$+198562592$
$+249124910$
$\overline{\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad}$
9930115350113787015625

4. $(2436)^2=\begin{array}{c}5761296\\ \hline 1728\\ \hline 5934096\end{array}$; $68\times132=8976$; $8852\times8948=79207696$; $868\times932=808976$.

5. $48\frac{17}{25}\times 49\frac{12}{25}=2400\frac{897}{841}$; $1295\frac{57}{63}\times 1296\frac{36}{63}=167761\frac{573}{864}\frac{43}{6}$.

6. $7464\times 7536=56248704$; $88044\times 87956=7744000000-1936$.

7. $2777\frac{7}{9}\times 41666\frac{2}{3}\times 666\frac{2}{3}\times 54\times 24\times 52\times 7692307\frac{9}{13}\times 625\times 125\times 56\times 32\times 1428571428571428\frac{4}{7}\times 2083\frac{1}{3}\times 48\times 833\frac{1}{3}\times 3125\times 68543764287590=\ldots$"

These and eleven other even longer computations are carried out mentally by Mr. Case.

The principle, $a^2=(a-b)(a+b)+b^2$, may be used in the squaring of any number, though it is not so readily used if the numbers consist of more than two digits. Thus,

$$87^2=(87-3)(87+3)+3^2=84\times 90+9,$$
$$92^2=(92-2)(92+2)+2^2=90\times 94+4.$$

This is the principle used in several of Mr. Case's calculutions. Thus, $(5\frac{1}{2})^2=(5\frac{1}{2}-\frac{1}{2})(5\frac{1}{2}+\frac{1}{2})+(\frac{1}{2})^2=30\frac{1}{4}$. ED. F.

330. Proposed by R. D. CARMICHAEL, Princeton, N. J.

An important function in the Theory of Numbers is one defined thus: $f(x)=1$ when $x>0$, $f(x)=0$ when $x=0$, $f(x)=-1$ when $x<0$. Two analytic expressions for $f(x)$ are the following:

$$f(x)=\lim_{n=\infty} x^{1/(2n-1)},\ n=1, 2, \ldots;\quad f(x)=\lim_{n=\infty}\frac{(x+1)^n-(x+1)^{-n}}{(x+1)^n+(x+1)^{-n}},\ x>-1.$$

It is required to find other non-trigonometric analytic expressions for this function. (There are several representations of $f(x)$ by means of trigonometric functions.)

Remark by the PROPOSER.

Professor F. H. Safford, of the University of Pennsylvania, has sent me the following expressions for the function defined in the problem:

$$\frac{2}{\pi}\int_0^\infty \frac{\sin xz}{z}dz,\quad \frac{2}{\pi}\int_0^\infty \frac{xdz}{x^2+z^2},\quad \operatorname{Lim}_{m=+\infty}\frac{e^{xm}-e^{-xm}}{e^{xm}+e^{-xm}}.$$

333. Proposed by R. D. CARMICHAEL, Princeton University.

Sum the infinite series

$$\frac{1}{(m+1)^2}+\frac{(2m-1)}{(2m+1)^3}+\frac{(3m-1)^2}{(3m+1)^4}+\frac{(4m-1)^3}{(4m+1)^5}+\frac{(5m-1)^4}{(5m+1)^6}+\ldots$$

[No solution of this problem has been received.]

334. Proposed by G. B. M. ZERR. A. M., Ph. D., Philadelphia, Pa.

Sum the series, $2^n-n\,2^{n-2}+\frac{n(n-3)}{2!}2^{n-4}-\frac{n(n-4)(n-5)}{3!}2^{n-6}$
$+\frac{n(n-5)(n-6)(n-7)}{4!}2^{n-8}-\frac{n(n-6)(n-7)(n-8)(n-9)}{5!}2^{n-10}+\ldots$

Solution by the PROPOSER.

We have, $(1-px)(1-qx) \equiv 1-x(p+q-pqx)$. Taking logarithms,

$$\log(1-px)+\log(1-qx)=\log[1-x(p+q-pqx)].$$

Hence, $(p+q)x+\frac{(p^2+q^2)x^2}{2}+\frac{(p^3+q^3)x^3}{3}+\ldots+\frac{(p^n+q^n)x^n}{n}$

$$=x(p+q-pqx)+\frac{x^2(p+q-pqx)^2}{2}+\ldots+\frac{x^n(p+q-pqx)^n}{n}.$$

The coefficients of x^n are equal for values of x, which makes the series convergent.

$$\therefore \frac{p^n+q^n}{n}=\frac{(p+q)^n}{n}-\frac{(n-1)pq(p+q)^{n-2}}{n-1}$$

$$+\frac{(n-2)(n-3)p^2q^2(p+q)^{n-4}}{2!(n-2)}-\frac{(n-3)(n-4)(n-5)p^3q^3(p+q)^{n-6}}{3!(n-3)}+\ldots$$

Hence, $p^n+q^n=(p+q)^n-npq(p+q)^{n-2}$

$$+\frac{n(n-3)}{2!}p^2q^2(p+q)^{n-4}-\frac{n(n-4)(n-5)}{3!}p^3q^3(p+q)^{n-6}+\ldots$$

Let $p=q$. Then $2=2^n-n2^{n-2}+\frac{n(n-3)}{2!}2^{n-4}-\frac{n(n-4)(n-5)}{3!}2^{n-6}$

$$+\frac{n(n-5)(n-6)(n-7)}{4!}2^{n-8}-\ldots$$

Solved similarly by V. M. Spunar, who starts with the identity, $2\log(1-x)=\log(1-2x+x^2)$.

GEOMETRY.

359. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

Two tangents are drawn to two confocal parabolas from any point on a common tangent. Show that the former two tangents and their chord of contact envelop yet another confocal parabola.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let S be the focus, T the point on the common tangent; TP, TQ the tangents to one parabola; TR, TU the tangents to the other parabola; TP, TR being the common tangent. Then S, P, Q, R, U, T are concyclic.

Hence, since this circle passes through S, T, Q, U, it passes through the intersections T, Q, U of the three lines TQ, TU, QU.

$\therefore$ The parabola that is tangent to TQ, TU, QU has S for its focus. (Todhunter, *Conic Sections*, Art. 146.)

II. Solution by S. LEFSEHETZ, East Pittsburg, Pa.

First solution [Fig. 1]. Let F be the common focus, T_1 and D_1, T_2 and D_2 the tangents and directrices of the given parabola; M a point on the common tangent; MC_1 and MC_2 the two tangents drawn by ordinary construction to the two parabolas; C_1 and C_2 their contacts; C_1a_1 and C_2a_2 the perpendiculars drawn from them to the respective directrices; P_1 and P_2 their intersection with T_1 and T_2. By well known properties of parabolas we know that FP_1 is perpendicular to MP_1.

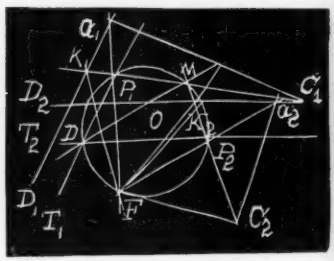


Fig. 1.

$\therefore \angle FP_1T_1 = \angle a_1C_1P_1 = \angle P_1C_1F$.

Also, $\angle FP_2T_2 = \angle P_2C_2F$. Now MC, the common tangent, passes through the point where T_1 and T_2 meet, since FD perpendicular to MC must meet it on both these lines. Therefore in circle O (drawn on MF as diameter) $\angle DP_1F = \angle DP_2F = \angle P_1C_1F = \angle P_2C_2F$.

$\therefore \angle MC_1C_2 = \angle FC_1C_2 = \angle K_1FK_2$ since K_1F and K_2F are respectively perpendicular to FC_1 and FC_2.

$\therefore$ Circles circumscribed to triangles K_1MK_2 and C_1MC_2 meet both in F, and if we apply Simpson's theorem, we find that the perpendiculars drawn from F to MC_1, MC_2, K_1K_2, C_1C_2 have their feet on a straight line. Therefore F is the focus of a parabola touching these four lines—one more than required.

Second solution. Transform by reciprocal polars, taking F for center of transformation (Fig. 2). Then our conics become circles O and O_1 meeting in F and in another point A. We have now to prove that MN being a

common chord of circles O and O_1, passing through A, MC and NC, respective tangents in M and N, B point where OM and O_1N meet, the five points F, B, M, C, N, are on the same circle. That B, M, N. C are on the same circle is evident. Then $\angle AFM = \angle CMA$, $\angle AFN = \angle ANC$.

$\therefore \angle MFN = \angle CMA + \angle CNA = 180° - \angle C$, which proves that F is on circle $BMNC$, and therefore the proposition.

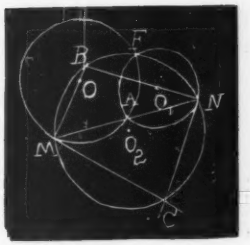


Fig. 2.

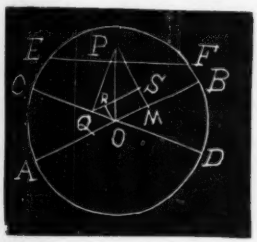
This is in all its generality, for we can readily see that B is the transformed of K_1K_2—line joining points where the tangents meet the respective directrices.

360. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

A circular segment, area A, revolves successively about the diameters (fixed) d, d', intersecting at an angle θ. If v = volume about d, v' the volums about d', then $v^2 + v'^2 - 2vv' \cos\theta$ is independent of the position of the segment.

Solution by S. LEFSEHETZ, East Pittsburg, Pa., and the PROPOSER.

Let P be the center of gravity of the segment; EF, its chord; O the center of the circle; and AB, CD, the diameters d, d', respectively; $\angle AOC = \angle BOD = \theta$.



Draw PQ perpendicular to CD, PM perpendicular to AB, QS perpendicular to PM, and OR perpendicular to QS.

Let $PQ = a$, $PO = c$, $PM = b$.

Then $b = PS + SM = PS + OR = a\cos\theta + \sqrt{(c^2 - a^2)}\sin\theta$. $v = 2\pi Ab$, $v' = 2\pi Aa$.

$\therefore v^2 + v'^2 - 2vv'\cos\theta = 4\pi^2A^2(a^2 + b^2 - 2ab\cos\theta) = \triangle$.

$\therefore \triangle = 4\pi^2A^2[a^2 + a^2\cos^2\theta + 2a\sin\theta\cos\theta\sqrt{(c^2 - a^2)} + (c^2 - a^2)\sin^2\theta - 2a^2\cos^2\theta - 2a\sin\theta\cos\theta\sqrt{(c^2 - a^2)}]$.

Hence, $\triangle = 4\pi^2A^2c^2\sin^2\theta$.

Solved similarly by S. G. Barton, J. Scheffer, and A. H. Holmes.

CALCULUS.

Remark on 282, by F. H. SAFFORD, Ph. D., University of Pennsylvania.

The published solution of 282 is incomplete. It is valid for a *long* box, but with a *short* box, the outer corner, B, may not reach its maximum before the corner, A, in Dr. Zerr's figure *emerges* from the hall. See Fig. 2, page 186, October, 1907. In that figure, N is omitted but should be vertically *above* Q. S is the corner vertically *under* M. In line 2, page 187, omit b preceding the word, becomes.

289. Proposed by G. W. DROKE, Professor of Mathematics, University of Arkansas.

Find the curve such that the rectangle under the perpendiculars from two fixed points on the normals be constant.

Solution by S. G. BARTON, Ph. D., Clarkson School of Technology, Potsdam, N. Y.

This is problem 27, page 61, Murray's *Differential Equations*.

Take the straight line through the fixed points as the x axis, and the middle point as the origin. The coordinates of the points are then $(a, 0)$ and $(-a, 0)$. Using p for $\frac{dy}{dx}$, the equation of a normal at (x', y') is

$$y-y'=-\frac{1}{p}(x-x').$$

Its distance from $(a, 0)$ is

$$\frac{\frac{a}{p}-\left(y'+\frac{x'}{p}\right)}{\sqrt{1+\frac{1}{p^2}}},$$

and its distance from $(-a, 0)$ is

$$\frac{\frac{-a}{p}-\left(y'+\frac{x'}{p}\right)}{\sqrt{1+\frac{1}{p^2}}}.$$

The constant product, dropping primes, is

$$\frac{y^2+\frac{2xy}{p}+\frac{x^2}{p^2}-\frac{a^2}{p^2}}{1+\frac{1}{p^2}}=k^2,$$

or $p^2y^2+2xyp+x^2-a^2=k^2p^2+k^2$, or $(y^2-k^2)p^2+2xyp+x^2-a^2-k^2=0$;

whence the p discriminant is

$$4x^2y^2-4(y^2-k^2)(x^2-a^2-k^2).$$

Hence, $$4x^2y^2-4x^2y^2+4(k^2+a^2)y^2-k^2(x^2-a^2-k^2)=0,$$

$$\text{or, } \frac{y^2}{k^2}+\frac{x^2}{k^2+a^2}=1.$$

This is the singular solution of the differential equation. It represents a system of confocal conics having the fixed points as foci.

Also solved by V. M. Spunar, G. B. M. Zerr, and J. Scheffer.

Solution by PROFESSOR F. L. GRIFFIN, Williams College.

However the ellipse is placed relative to the co-ordinate axes, its area is

$$S=\int_{x_1}^{x_2}(y''-y')\,dx,$$

where x_1 and x_2 $[x_1<x_2]$ are the extreme abscissas taken in the curve, and y' and y'' $[y'\leq y'']$ are the two ordinates corresponding to any one value of x. Solved for y, the given equation becomes

$$by=-(hx+f)\pm\sqrt{[f^2-bc+2(hf-bg)x+(h^2-ab)x^2]}.$$

Now let

$$f^2-bc=b^2A,\quad fh-bg=b^2B,\quad ab-h^2=b^2C,$$

where for an ellipse $C>0$. Then the difference of the two ordinates becomes

$$y''-y'=2\sqrt{[A+2Bx-Cx^2]}.$$

Hence, integrating,

$$S=2\left[\frac{Cx-B}{2C}\sqrt{[A+2Bx-Cx^2]}+\frac{B^2+AC}{2C^{\frac{3}{2}}}\sin^{-1}\frac{Cx-B}{\sqrt{[B^2+AC]}}\right]_{x_1}^{x_2}.$$

Now the extreme abscissas make $y'=y''$, or $A+2Bx-Cx^2=0$; whence

$$Cx_2=B+\sqrt{[B^2+AC]}\text{ and }Cx_1=B-\sqrt{[B^2+AC]}.$$

Substituting these values,

$$S=\frac{B^2+AC}{C^{\frac{3}{2}}}[\sin^{-1}1-\sin^{-1}(-1)]=\frac{\pi}{b}\frac{[(fh-bg)^2+(ab-h^2)(f^2-bc)]}{(ab-h^2)^{\frac{3}{2}}},$$

which immediately reduces to the formula proposed.

Also solved by G. B. M. Zerr and J. Scheffer.

MECHANICS.

240. Proposed by S. A. COREY, Hiteman, Iowa.

A perfectly flexible wire rope weighing one pound per foot is suspended from the tops of two vertical supports 300 feet apart, one support being 30 feet higher than the other. One end of the rope is fastened to the top of the higher support, while 600 feet of the rope hangs vertically from the top of the lower support. Assuming that the rope is free to slide over the top of the lower support without friction, find the lowest point of

that portion of the rope which is suspended between the supports. Also find the amount of work which must be performed in raising the lowest point to make it coincide with the top of the lower support by exerting a pull on the free end of the rope.

[No solution of this problem has been received.]

240. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

A simple beam length $2a$, supported at both ends, is loaded in the form of a parabola, height of vertex b. Find deflection at center due to this load.

Solution by the PROPOSER.

Let AB be the beam, ACD the parabola, $CD=b$, $AD=DB=a$. Take E any point on AB, draw EF perpendicular to AB. Let $AE=x$, and also z be the distance of the center of gravity of the area AEF from EF. Then $\frac{2}{3}ab$=total load. $(x-a)^2+(a^2/b)(y-b)=0$ is the equation to the parabola, with A as origin.

$$\text{Then } (x-z)=\frac{\int\int x\,dx\,dy}{\int\int dx\,dy}=\frac{\int\int x\,dx\,dy}{A}.$$

$$\therefore A(x-z)=\int_0^x xy\,dx=\frac{b}{a^2}\int_0^x(2ax^2-x^3)\,dx=\frac{bx^3}{12a^2}(8a-3x).$$

$$A=\int_0^x y\,dx=\frac{bx^2}{3a^2}(3a-x).$$

$$\therefore x-z=\frac{8ax-3x^2}{4(3a-x)} \text{ and } z=\frac{4ax-x^2}{4(3a-x)}.$$

Taking moments about A we get,

$$EI\frac{d^2y}{dx^2}=\tfrac{2}{3}abx-Az=\tfrac{2}{3}abx-\frac{bx^3}{3a}+\frac{bx^4}{12a^2}=M.$$

$$EI\frac{dy}{dx}=\tfrac{1}{3}abx^2-\frac{bx^4}{12a}+\frac{bx^5}{60a^2}+C.$$

When $x=a$, $dy/dx=0$, $C=-\frac{4}{15}a^3b$.

$$\therefore EI\frac{dy}{dx}=\tfrac{1}{3}abx^2-\frac{bx^4}{12a}+\frac{bx^5}{60a^2}-\tfrac{4}{15}a^3b,$$

$$EIy=\tfrac{1}{6}abx^3-\frac{bx^5}{60a}+\frac{bx^6}{360a^2}-\tfrac{4}{15}a^3bx=-\frac{61a^4b}{360} \text{ when } x=a.$$

$$\therefore y=-\frac{61a^4b}{360EI}=\text{deflection required.}$$

For cantilever beam, length $2a$, with the same load,

$$EI\frac{d^2y}{dx^2}=-A(x-z)=-\frac{2bx^3}{3a}+\frac{bx^4}{4a^2},$$

$$EI\frac{dy}{dx}=-\frac{bx^4}{6a}+\frac{bx^5}{20a^2}+C;\ \text{when } x=2a,\ dy/dx=0,\ C=\tfrac{16}{15}a^3b.$$

$$\therefore EI\frac{dy}{dx}=-\frac{bx^4}{6a}+\frac{bx^5}{20a^2}+\tfrac{16}{15}a^3b.$$

$$EIy=-\frac{bx^5}{30a}+\frac{bx^6}{120a^2}+\tfrac{16}{15}a^3bx=\tfrac{24}{15}a^4b,\ \text{when } x=2a.$$

$$\therefore y=\frac{24a^4b}{15\,EI}=\text{deflection at end of beam.}$$

Also solved by Harold Rowe.

242. Proposed by C. N. SCHMALL, 604 East 5th Street, New York City.

In a certain New York theatre there is an asbestos curtain supported by thin circular rings, radius r, which move on a cylindrical rod of radius a. The curtain is intended to be drawn by a *steady pull*. Taking μ as the coefficient of friction, show that this will not be possible if r be less than $a\sqrt{(1+\mu^2)}$.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let P be the resultant pull on a ring, θ the angle between the direction of P and the normal to the surface of contact of ring and rod.

Then $P\cos\theta$=resolved part of P along the normal, and $P\sin\theta$=resolved part at right angles to the normal. For equilibrium, $P\sin\theta<\mu P\cos\theta$.

$$\therefore \sin^2\theta<\mu^2\cos^2\theta,\ \text{or}\ \cos^2\theta>\frac{1}{1+\mu^2}.$$

The diameter of the ring is in the direction of P; diameter of rod is the normal.

$$\therefore \cos\theta=\frac{a}{r}.\quad \therefore\frac{a^2}{r^2}>\frac{1}{1+\mu^2}\ \text{or}\ r<a\sqrt{(1+\mu^2)}.$$

243. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

A weight W is supported by three strings of the same size and quality lying in the same plane. The middle string is vertical, one string makes with it an angle θ on one side, and the other string makes with it an angle ϕ on the other side. Find the stresses T_1, T_2, T_3 in the strings.

Solution by the PROPOSER.

Let AQ be the vertical string; BQ, CQ, the strings making angles θ, ϕ with AQ, respectively. Let $AQ=a$, $BQ=b$, $CQ=c$, E=Young's modulus, β=the sectional area of string, elongation of $AQ=a_1$, of $BQ=b_1$, of $CQ=c_1$; T_1 the stress in AQ; T_2 the stress in BQ; T_3 the stress in CQ.

Then $T_1+T_2\cos\theta+T_3\cos\phi=W$... (1).

$T_2\sin\theta=T_3\sin\phi$... (2).

$T_1a_1+T_2b_1+T_3c_1=$minimum... (3).

But $a_1=aT_1/(E\beta)$, $b_1=bT_2/(E\beta)$, $c_1=cT_3/(E\beta)$. Hence (3) becomes

$$(T_1^2a+T_2^2b+T_3^2c)/(E\beta)=\text{minimum}.$$

Now $a=b\cos\theta=c\cos\phi$.

$$\therefore \frac{a}{E\beta}\left[T_1^2+\frac{T_2^2}{\cos\theta}+\frac{T_3^2}{\cos\phi}\right]=\text{minimum}... (4).$$

From (1), $dT_1+dT_2\cos\theta+dT_3\cos\phi=0$... (5).

From (2), $dT_2\sin\theta=dT_3\sin\phi$... (6).

From (4), $T_1dT_1+(T_2/\cos\theta)dT_2+(T_3/\cos\phi)dT_3=0$... (7).

Eliminating dT_1, dT_2, dT_3 between (5), (6), (7), we get

$$T_1\cos\theta\sin(\theta+\phi)=T_2\sin\phi\cos\phi+T_3\sin\theta\cos\theta... (8).$$

From (1), (2), and (8),

$$T_1=\frac{W(\sin^2\phi\cos\phi+\sin^2\theta\cos\theta)}{\sin^2\phi\cos\phi+\sin^2\theta\cos\theta+\cos\phi\cos\theta\sin^2(\theta+\phi)},$$

$$T_2=\frac{W\sin\phi\cos\phi\cos\theta(\sin\phi\cos\phi+\sin\theta\cos\theta)}{\sin^2\phi\cos\phi+\sin^2\theta\cos\theta+\cos\phi\cos\theta\sin^2(\theta+\phi)},$$

$$T_3=\frac{W\sin\theta\cos\theta\cos\phi(\sin\phi\cos\phi+\sin\theta\cos\theta)}{\sin^2\phi\cos\phi+\sin^2\theta\cos\theta+\cos\phi\cos\theta\sin^2(\theta+\phi)}.$$

If $\theta=\phi$, $T_1=\frac{W}{(1+2\cos^3\theta)}$. $T_2=T_3=\frac{W\cos^2\theta}{(1+2\cos^3\theta)}$.

(See problems 186d, 186e, page 476, Merriman's *Mechanics of Materials.*)

244. Proposed by C. N. SCHMALL, New York City.

In a game of billiards a player observes two balls, A and B, at rest in a certain position and concludes that it would be to his advantage to project A against B in such a manner that, as a result of the impact, A might suffer the greatest possible deviation from its course. Taking the balls to be equal and smooth, each of diameter a and elasticity e, and the distance between their centers to be d, show that he can accomplish the desired motion by projecting A in a line making an angle equal to the

$$\sin^{-1}\frac{a}{d}\sqrt{\frac{1-e}{3-e}}$$

with the line joining the centers.

Solution by S. G. BARTON, Ph. D., Clarkson School of Technology.

Let α be the angle between the direction of the motion of the impinging ball and the line joining the centers at the time of impact, and x the required angle. We then have, from the law of sines,

$$\frac{\sin x}{\sin\alpha}=\frac{a}{d},\ \text{or}\ x=\sin^{-1}\left[\frac{a}{d}\sin\alpha\right].$$

When a moving smooth ball strikes another smooth ball of the same weight at rest, we have that the tangent of the angle between its original and new directions is (*c. f.* Bowser's *Analytic Mechanics*, problem 30, page 387),

$$\frac{1+e}{2}\cdot\frac{\sin 2\alpha}{1+\sin^2\alpha-e\cos^2\alpha}.$$

The deviation will be a maximum when its tangent is a maximum. Neglecting the constant, and differentiating for the maximum, we find the condition to be

$$\begin{aligned}&[1+\sin^2\alpha-e\cos^2\alpha]2\cos 2\alpha-\sin^2 2\alpha[1+e]=0,\\ \text{or, }&2[(1-e)+(1+e)\sin^2\alpha]\ [1-2\sin^2\alpha]-4\sin^2\alpha[1-\sin^2\alpha]\ [1+e]=0,\\ \text{or, }&2[1-e]-2[3-e]\sin^2\alpha=0;\end{aligned}$$

whence for the maximum deviation,

$$\sin\alpha=\sqrt{\frac{1-e}{3-e}},\ \text{and}\ x=\sin^{-1}\frac{a}{d}\sqrt{\frac{1-e}{3-e}}.$$

Also solved by G. B. M. Zerr.

NOTES AND NEWS.

The fourth article in the series on the teaching of collegiate mathematics is found in this number, namely: The Teaching of Calculus, by Professor W. B. Ford, of the University of Michigan. Intimately related to this article is a review of an English text on the Calculus by Dr. N. J. Lennes of the Massachusetts Institute of Technology. The fifth article of the series comes next month by Professor Frederick H. Bailey of the Massachusetts Institute of Technology on Unified Mathematics.

Prof. Leonard E. Dickson, formerly editor of the MONTHLY, has just returned to residence at the University of Chicago after a leave of absence which he spent abroad. He has been elected Chairman of the Chicago section of the American Mathematical Society for the present year.

Dr. N. J. Lennes, of the Massachusetts Institute of Technology, has received an appointment as instructor in mathematics at Columbia University, beginning in the autumn of 1910. During the summer he will be in charge of the mathematical courses at the Chautauqua Summer School, Chautauqua, N. Y.

At the spring meeting of the Chicago section of the American Mathematical Society, held at the University of Chicago, April 8-9, 1910, seventeen papers were read and about fifty members were in attendance.

The demand for trained teachers of mathematics for colleges and universities seems far in excess of the supply. At least such is the report from the larger universities which turn out such teachers. The present year is no exception. Already the calls are numerous and urgent. The word would seem to be that more young men should take advantage of the opportunity and at once enter upon a course which will provide the adequate training to meet such demands.

Mr. Egbert J. Miles has been appointed to an instructorship in mathematics at Cornell University, Ithaca, N. Y.

Mr. Arthur D. Pilcher, who has been on leave of absence from the University of Kansas during the past two years, will return to his work in the department of mathematics there with the opening of the autumn semester, 1910.

Professor Archibold Henderson, of the University of North Carolina, has been granted a leave of absence for the year 1910-11.

Professor E. J. Wilczynski, of the University of Illinois, will give courses in mathematics at the University of Chicago during the summer quarter, 1910.

Dr. Anthony L. Underhill, of the University of Minnesota, has been appointed instructor for the summer quarter, 1910, at the University of Chicago.

Miss E. R. Bennett, Fellow at the University of Illinois, has been appointed instructor in mathematics at the University of Kansas.

At the summer session at the University of Illinois, the following courses in mathematics will be offered: Advanced Algebra, Plain Trigonometry; Teachers' Course for Secondary Teachers, Plain Analytic Geometry, Differential Calculus, Integral Calculus, Differential Equations, Elementary Theory of Groups, Theory of Equations and Determinants, Seminar and Thesis Course.

Can any of our subscribers furnish us a copy of No, 1, Vol. XI of MONTHLY. Also a copy each of Nos. 6 and 11 of Vol II? We have calls for these numbers and we are willing to pay a fair price for them. F.

Professor Oscar Bolza has resigned at the University of Chicago and will return to Germany, where he will lecture in the University of Freiburg. He has been identified with the University of Chicago since its foundation in 1892, and his loss will be felt by a large number of students who love him for his kindly friendship and his inspiring teaching, as well as admire him for his distinguished scholarship in his chosen field of research. Included in this body of his students are no less than seven of his present colleagues at Chicago, who will feel his loss even more keenly than those who are situated in other parts of the country.

The Central Association of Science and Mathematics Teachers is emphasizing the concrete presentation in all branches, and in this spirit the mathematics section has been collecting, through a committee, an extended list of real applied problems in algebra and geometry, from a variety of sources. These have been published in *School Science and Mathematics* during the past two years and have now been gathered and classified in a small pamphlet and are being tried out by a large number of teachers. Copies of this pamphlet may be secured in any quantity, at five cents each, from the secretary of the Mathematical Section, Miss Mabel Sykes, 1223 East 57th street, Chicago, Ill.

Dr. George Bruce Halsted has the work of revising the definitions of the mathematical terms and the selection of new mathematical words for the two new volumes of the *Century Dictionary* which are now going through the press. Some of the new words introduced in current mathematical literature have been coined by Dr. Halsted. F.

"Periodic Orbits about an Oblate Spheroid" is the title of an extended article in the January, 1910, number of the *Transactions of the American Mathematical Society*, by Dr. W. D. MacMillan of the University of Chicago.

Mr. E. B. Escott, instructor in mathematics at the University of Michigan, has an article on "Logarithmic Series" in the *Quarterly Journal of Pure and Applied Mathematics*, No. 162, 1910, and also an article on "Cubic Congruences with Three Real Roots" in the *Annals of Mathematics* for January, 1910.

Dr. E. J. Wilczynski, of the University of Illinois, has been appointed to an associate professorship in mathematics at the University of Chicago. He will enter upon his work at the beginning of the Summer Quarter, 1910.

The Summer Quarter, 1910, at the University of Chicago, will begin Monday, June 20. The courses in mathematics include Trigonometry, College Algebra, Synopic Course in Pure and Applied Mathematics, Differential Calculus, Integral Calculus, Theory of Equations, Differential Equations, Critical Review of Secondary Mathematics, Graphical Methods in Algebra, Theory of Substitutions, General Analysis, Functions of a Complex Variable, Modern Analytic Geometry, Projective Differential Geometry, Seminar on the Foundations of Mathematics, and Reading and Research in Pure Mathematics. The first term extends to July 27, and the second term to September 2.

The courses in advanced mathematics at the various American and foreign universities are announced with some regularity in the *Bulletin of the American Mathematical Society*. Those for 1910—1911 at Cornell, Princeton, Yale, and the University of Strassburg are found in the May, 1910, issue. In this same number also is a list of the doctorates for the year 1908—1909 conferred by the various German universities.

Mr. A. S. Hawkesworth has been appointed professor of higher mathematics in the University of Pittsburgh, Pittsburgh, Pennsylvania.

This issue was mailed May 21.

ERRATA.

The statement of the following problem should appear at the top of page 95.

290. Proposed by C. N. SCHMALL, New York City.

When the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$, represents an ellipse, show (by integration) that its area is

$$\frac{\pi\,(af^2+bg^2+ch^2-abc-2fgh)}{(ab-h^2)^{\frac{3}{2}}}.$$

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UNIFIED MATHEMATICS.

By F. H. BAILEY, Massachusetts Institute of Technology.

One of the first questions which a student asks at, or near, the beginning of his study of any branch of mathematics is, "What is it good for?" In most cases he is not satisfied unless he can be shown some specific utilitarian use to which his new knowledge may be put. With many students this attitude of mind persists throughout their study, which, for lack of a satisfying answer, is often closed at an earlier date than is desirable. Other students continue their mathematics solely for the use they can make of it as a tool in their study of some other science, as physics, for example.

Good pedagogy requires that the solution of concrete problems should form a large and important part of any course of mathematics, not merely because it will increase the student's interest in his work and at the same time give him the ability to use his mathematics as a tool, but because it will aid him in acquiring the power of close and accurate reasoning, of careful analysis, and of seeing the mathematical relationships of quantities — in a word — mathematical power. But with this end in view the mere results of the problems, which the utilitarian type of mind would prize so highly, become relatively insignificant, and the methods of attack and solution are of chief importance. In fact the extreme utilitarian, omitting everything which does not seem directly applicable to some "practical" problem, and in what he retains, focusing his attention upon the results, will defeat his own ends. For his students will be left with a few rules applicable in the particular cases studied, but they will have little or no mathematical ability— no power of analysis and no general theory for attacking new types of problems.

It must be acknowledged, however, that a student of good mathematical ability, or one who has been well taught and has followed a well arranged program of mathematical study, oftentimes is apparently unable to apply his mathematics to the study of a new science — in other words, he seems to lack that mathematical habit of mind at which we have aimed. In many cases, however, this failure is only apparent and the real difficulty is